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Analyze of The Distance k-Domination Number of The Amalgamation of Complete and Star Graph

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Abstrak. Penelitian ini meneliti konsep bilangan dominasi jarak-k (distance k-domination number) dengan secara khusus mengkaji penerapannya pada graf amalgamasi pada graf lengkap dan graf bintang. Graf amalgamasi, yang dinotasikan sebagai $Amal(G, v, n)$, adalah graf yang dibangun dari graf dasar G, sebuah simpul tertentu v di G, dan bilangan bulat positif n. Graf amalgamasi dibentuk dengan menyisipkan n salinan dari graf G pada simpul v, di mana semua simpul v dalam n salinan tersebut digabungkan menjadi satu titik. Bilangan dominasi jarak-k adalah kardinalitas minimum dari himpunan dominasi jarak-k, yang dinotasikan sebagai $\gamma_k(G)$. Melalui formulasi matematika dan prinsip-prinsip teori graf, kami menetapkan sifat-sifat dan batasan bilangan dominasi jarak-k pada amalgamasi graf lengkap (K_n) dan graf bintang (S_n).

Kata kunci: himpunan dominasi jarak k, bilangan dominasi jarak k, amalgamasi, graf lengkap, graf bintang.

Abstract. *This paper explores the concept of the distance k-domination number specifically examining its application to the amalgamation of complete graphs and star graphs. The amalgamation graph denoted by $Amal(G, v, n)$ is a graph constructed from a given base graph G, a specified vertex v in G, and a positive integer n. The amalgamation graph is formed by attaching n copies of the graph G at the vertex v, merging all the v-vertices in the n copies into a single vertex. The distance k-domination number is the minimum cardinality of the distance k-dominating set, denoted by $\gamma_k(G)$. Through mathematical formulations and graph-theoretical principles, we establish the properties and bounds of the distance k $-$domination number on amalgamation of complete graph (K_n) and star graph (S_n).*

Keywords: *distance-k dominating set, distance-k domination number, amalgamation, complete graph, star graph.*

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[1] INTRODUCTION

Graph theory plays a pivotal role in modeling and analyzing complex networks in various disciplines, including computer science, biology, transportation, and social networks [1]. Among the numerous parameters studied in graph theory, the distance k-resolving domination number emerges as a powerful tool that combines the concepts of domination and metric dimension in a unified framework.

A dominating set in a graph is a set of vertices such that every vertex in the graph is either in the set or adjacent to at least one vertex in the set. Simultaneously, a resolving set uniquely identifies all vertices in the graph based on their distances from the set's members [2]. The definition and the latest result about distance k dominating set in graph, can be seen at [3]-[8]. The distance k-resolving domination number generalizes these notions by considering vertices within a distance k and requiring the set to both dominate and resolve the graph in this localized neighborhood. This parameter is significant in applications such as network navigation, fault diagnosis, and resource allocation in distributed systems.

In this study, we focus on the amalgamation of graphs, which is a construction method where multiple graphs are joined by identifying certain vertices or edges. The amalgamation process is widely used to create larger networks from smaller graph components, allowing for the analysis of complex systems. This paper aims to investigate how the distance k-resolving domination number behaves under graph amalgamation.

An amalgamation graph, denoted as $Amal(G, v, n)$, is a graph constructed from a given base graph G, a specified vertex v in G, and a positive integer n. The amalgamation graph is formed by attaching n copies of the graph G at the vertex v, merging all the v-vertices in the n copies into a single vertex [10].

Let G be a graph and $v \epsilon V(G)$ a vertex in G. The amalgamation graph $Amal(G, v, n)$ is constructed as follows :

1. Create n disjoint copies of G, say $G_1, G_2, \ldots G_n$, with corresponding vertex sets $V(G_1), V(G_2), \ldots, V(G_n)$ and edge sets $E(G_1), E(G_2), \ldots, E(G_n)$.
2. Identify (merge) the vertices corresponding to v in $G_1, G_2, \ldots, G_n$ into a single vertex $v\star$

The resulting graph $Amal(G, v, n)$ has :

1. A vertex set $V(Amal(G, v, n))$ consisting of the union of the vertex sets of $G_1, G_2, \ldots G_n$, with the v-vertices merged into $v\star$.
2. An edge set $E(Amal(G, v, n))$ consisting of the union of the edge sets of $G_1, G_2, \ldots G_n$.

$Amal(G, v, n)$ essentially creates a "star-like" structure with n branches, where each branch is a copy of G connected through the vertex $v\star$. The vertex $v\star$ acts as the central hub that unifies the n copies. This concept is used in graph theory to model scenarios where a graph structure is replicated and connected via a common central point. For an illustration, see Figure 1.

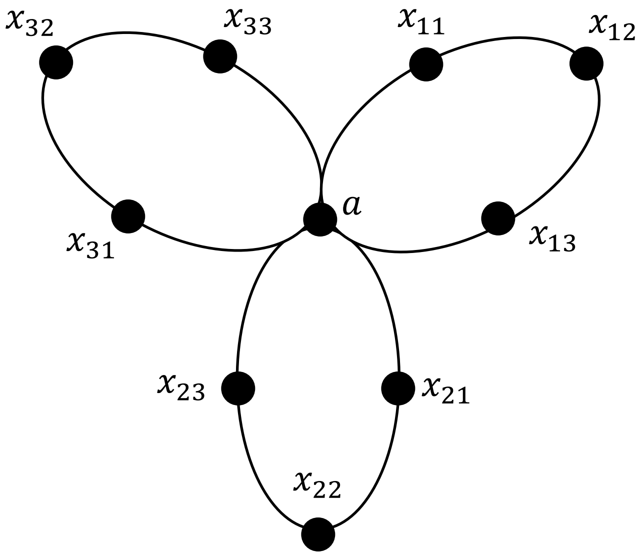


Figure 1. An illustration of $Amal(C_4, v, 3)$.

[2] RESULTS AND DISCUSSION

In this section, we show the result of the distance k domination number and the distance k resolving domination number of amalgamation of complete graph K_n and star graph S_n. We divided by two subsection, the first subsection is about distance k domination number of amalgamation graphs and the second subsection is about distance k resolving domination number of amalgamation graphs.

Bagian ini merupakan bagian terpenting dari sebuah isi artikel. Sitasi harus banyak dimunculkan pada bab ini. Dalam bagian ini merupakan jawaban dari sebuah tujuan penelitian. Selain dalam bentuk paragraf, bagian ini biasanya banyak memuat data-data pendukung berupa gambar, tabel, grafik, dll.

In this research we will focus on some special graphs, among others are: complete graph $Amal(K_n, v, m)$ and star graph $Amal(S_n, v, m)$.

The previous Results:

Lemma 1. [8] Let $diam(G) \leq 2$ and $k \geq 2$, so $\gamma_k(G) = 1$.

Distance k Domination Number of Amalgamation of Some Graphs

k domination number of amalgamation of complete graph K_n and star graph S_nThe complete graph on n vertices is denoted by K_n, K_n has $n(n-1)/2$ edges, $n-1$ degree and diameter K_n is 1 [9]. A star graph S_n is the complete bipartite graph $K_{1,n}$, a tree with one internal vertex and n leaves. Alternatively, some authors define S_n to be the tree of order n with maximum diameter 2; in which case a star of $n > 2$ has $n-1$ leaves [9].

Theorem 2. Let K_n be a complete graph, $n \geq 4$ and p is a terminal vertex of K_n. If G is the amalgamation of K_n, $Amal\{K_n, p, m\}$, p is a vertex as the center, then $\gamma_k(Amal(K_n, p, m)) = 1$.

Proof. Graph G is the amalgamation of complete graphs K_n, $n \geq 3$, $Amal(K_n, p, m)$. Every vertex of K_n can be a p because complete graph is a regular graph with $diam(K_n) = 1$. Every vertex of K_{n_i} have the distance 1 to p. So that, $diam(Amal(K_n, p, m)) = 2$.

Graph $Amal(K_n, p, m)$ has a vertex set $V\big(Amal(K_n, p, m)\big) = \{x\} \cup \{x_{i,j}\ ;\ 1 \leq i \leq\ n - 1;\ 1 \leq j \leq m\}$ and edge set $E\big(Amal(K_n, p, m)\big) = \big\{xx_{i,j}\ ;\ 1 \leq i \leq n-1;\ 1 \leq j \leq\ m\big\} \cup \{x_{i,j}x_{i,j}\ ;\ 1 \leq i \leq n-1\ ;\ 1 \leq j \leq\ m\}$. The cardinality of vertex set and edge set of $Amal(K_n, p, m)$ are $\big|V\big(Amal(K_n, p, m)\big)\big| = m(n-1) + 1, \big|E\big(Amal(K_n, p, m)\big)\big| = m\left(\frac{n(n-1)}{2}\right)$ respectively.

Based on **Lemma 1.**, graph with $diam(G) \leq 2$ has $\gamma_k(G) = 1$. Because $diam(Amal(K_n, p, m)) = 2$, so it is clear that $\gamma_k(Amal(K_n, p, m))\ =\ 1$. Choose $D = \{x\}$, because $x_{i,j}$ adjacent to x so every vertex $x_{i,j}$ dominated by x. For an illustration of Theorem 2, see Figure 2.

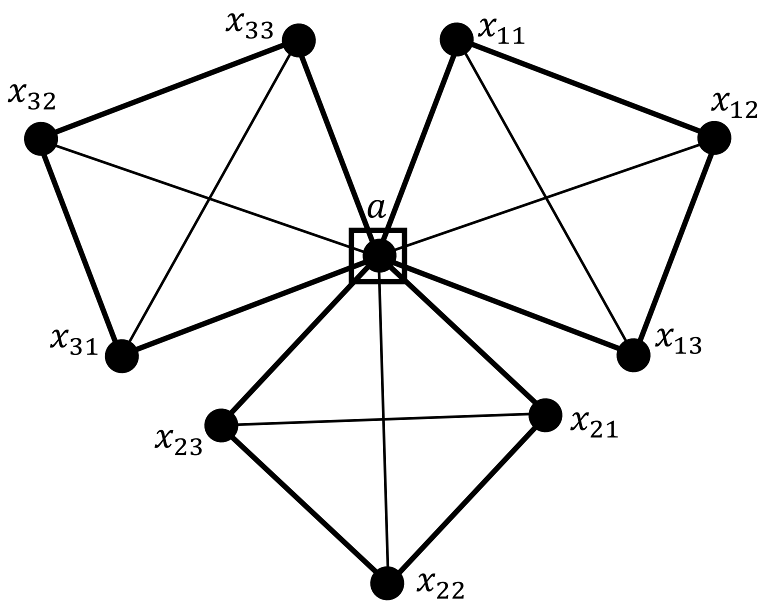


Figure 2. An illustration of distance k-dominating set with $k \geq 1$ of $Amal(K_4, v, 3)$.

Theorem 3. Let S_n be a star graph, $n \geq 3$. There are 2 types of vertex on star graph, central vertex (x) and non central vertex (x_i). If G is the amalgamation of S_n, $Amal\{S_n, p, m\}, p$ is a vertex as the center, then

$$\gamma_k\big(Amal(S_n, p, m)\big) = \begin{cases} 1, & for\ k \geq 1\ if\ x\ be\ a\ center\ and\ k \geq 2\ if\ x_i\ be\ a\ center, \\ m, & for\ k = 1\ if\ x_i\ be\ a\ center \end{cases}$$

Proof. Graph G is the amalgamation of star graph $(S_n, n \geq 3)$, $Amal(S_n, p, m)$. There are 2 types of vertex on star graph, central vertex (x) and non central vertex (x_i). There are 2 case for proof this Theorem.

Case 1. x is a vertex as the center

Graph $Amal(S_n, x, m)$ isomorphic with star graph S_n, so graph $Amal(S_n, x, m)$ has a vertex set $V\big(Amal(S_n, x, m)\big) = \{x\} \cup \{x_i;\ 1 \le i \le nm\}$ and edge set $E(Amal(S_n, x, m)) = \{xx_i;\ 1 \le i \le nm\}$. The cardinality of vertex set and edge set of $Amal(S_n, x, m)$ are $|V(Amal(S_n, x, m))| = nm + 1$, $|E(Amal(S_n, x, m))| = nm$ respectively. Because graph $Amal(S_n, x, m)$ isomorphic with star graph S_n, so $diam(Amal(S_n, x, m)) = diam(S_n) = 2$,

Based on Lemma 1, graph with $diam(G) \le 2$ has $\gamma_k(G) = 1$. Because $diam(Amal(S_n, x, m)) = 2$ so it is clear that $\gamma_k(Amal(S_n, x, m)) = 1$. Choose $D = \{x\}$, because x_i adjacent to x so every vertex x_i dominated by x. For an illustration, see Figure 3.

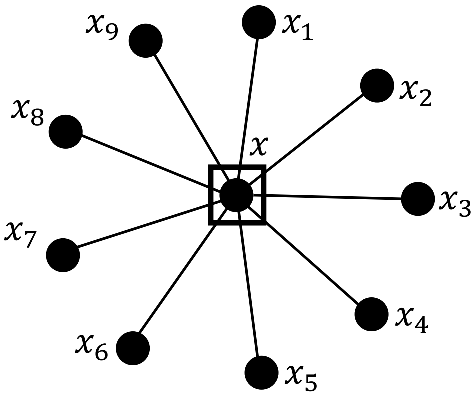


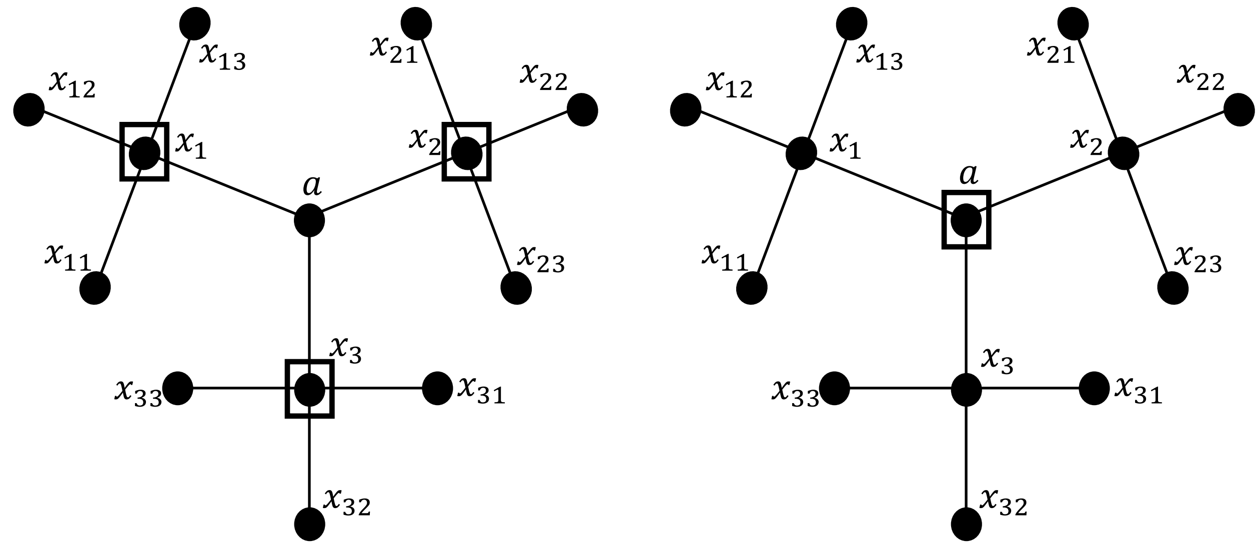
Figure 3. An illustration of distance k-dominating set with $k \ge 1$ of $Amal(S_3, x, 3)$.

Case 2. x_i is a vertex as the center.

Graph $Amal(S_n, x_i, m)$ has a vertex set $V\big(Amal(S_n, x_i, m)\big) = \{a\} \cup \{x_i;\ 1 \le i \le m\} \cup \{x_{i,j}\ ;\ 1 \le i \le m;\ 1 \le j \le n-1\}$ and edge set $E\big(Amal(S_n, x_i, m)\big) = \{ax_i;\ 1 \le i \le m\} \cup \{x_i x_{i,j}\ ;\ 1 \le i \le m;\ 1 \le j \le n-1\}$. The cardinality of vertex set and edge set of $Amal(S_n, x_i, m)$ are $|V(Amal(S_n, x_i, m))| = m + m(n-1) + 1$, $|E(Amal(S_n, x_i, m))| = m + m(n-1)$ respectively, $diam(Amal(S_n, x_i, m)) = 4$,

- For $k = 1$, Choose $D = \{x_i;\ 1 \le i \le m\}$, because vertex a and $x_{i,j}$ adjacent with x_i so $|D| = m$. $\gamma_k(Amal(S_n, x_i, m)) = m$.
- For $k \ge 2$ Choose $D = \{a\}$, because all the vertices have distance 2 to a, so $|D| = 1$. $\gamma_k(Amal(S_n, x_i, m)) = 1$.

For an illustration, see Figure 4.



[a] [b]

Figure 4. [a]. An illustration of distance k-dominating set with $k \geq 1$ of $Amal(S_3, x, 3)$ [b]. An illustration of distance k-dominating set with $k \geq 2$ of $Amal(S_4, x_i, 3)$.

[3] CONCLUSION

Based on the result in this paper, the exact value of the distance k-domination number of $Amal(K_n, p, m)$ is $\gamma_k\big(Amal(K_n, p, m)\big) = 1$ and $Amal(S_n, p, m)$ depends on the vertex p we choose, if we choose x to be a center, so $\gamma_k\big(Amal(S_n, x, m)\big) = 1$ for $k \geq 1$, if we choose x_i to be a center, so $\gamma_k\big(Amal(S_n, x_i, m)\big) = m$ for $k = 1$ and $\gamma_k\big(Amal(S_n, x_i, m)\big) = 1$ for $k \geq 2$. Therefore, this study found results that can be developed by raising open problems.

Open Problem 1. Determine the distance k-domination number of other graphs of $Amal(G, v, n)$!

Open Problem 2. Determine the characteristics of the distance k-domination number of $Amal(G, v, n)$!

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